

BANACH-ALAOGLU THEOREM FOR HILBERT H^* -MODULE

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ABSTRACT. We provided an analogue Banach-Alaoglu theorem for Hilbert H^* -module. We construct a Λ -weak* topology on a Hilbert H^* -module over a proper H^* -algebra Λ , such that the unit ball is compact with respect to Λ -weak* topology.

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1. Introduction

A Hilbert H^* -module L over an H^* -algebra Λ is a right Λ -module which possesses a $\tau(\Lambda)$ -valued product, where $\tau(\Lambda) = \{ab \mid a, b \in \Lambda\}$ is the trace-class. At the same time, H is a Hilbert space with the inner product given by the action of the trace on the $\tau(\Lambda)$ -valued product.

The notion of H^* -module is introduced by Saworotnow in [8] under the name of generalized Hilbert space. It has been studied by Smith [10], Giellis [4] Molnar [6], Cabrera [3], Martinez et al. [3], Bakić and Guljaš [2] and others. Saworotnow has proved that the trace-class in a proper H^* -algebra Λ has pre-dual. For Hilbert H^* -modules, a generalization of Riesz theorem holds, i.e. for each bounded Λ -linear functional on L , there is $x_f \in L$ such that $f(x) = [x_f, x]$ for all $x \in L$.

Paschke [7] showed that self-dual Hilbert W^* -modules are dual Banach spaces and found topology on module such that the unit ball is compact.

In the present paper we find a topology on Hilbert H^* -module L over a proper H^* -algebra Λ such that the unit ball in L is compact with respect to this topology.

2. Basic notations and preliminary results

We recall that an H^* -algebra is a complex associative Banach algebra with an inner product $\langle \cdot, \cdot \rangle$ such that $\langle a, a \rangle = \|a\|^2$ for all $a \in \Lambda$, and for each $a \in \Lambda$ there exists some $a^* \in \Lambda$ such that $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ba, c \rangle = \langle b, ca^* \rangle$ for all $b, c \in \Lambda$. The adjoint a^* of a need not be unique (see [1]). Proper H^* -algebra Λ is an H^* -algebra which satisfies $a\Lambda = 0 \Rightarrow a = 0$ (or $\Lambda a = 0 \Rightarrow a = 0$). An H^* -algebra Λ is proper if and only if each $a \in \Lambda$ has

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a unique adjoint $a^* \in \Lambda$ (see [1, Theorem 2.1]). An H^* -algebra is simple algebra if it has no nontrivial closed two-sided ideals.

The trace-class in a proper H^* -algebra Λ is defined as the set $\tau(\Lambda) = \{ab \mid a, b \in \Lambda\}$. The trace-class is selfadjoint ideal of Λ and it is dense in Λ , with norm $\tau(\cdot)$. The norm τ is related to the given norm $\|\cdot\|$ on Λ by $\tau(a^*a) = \|a\|^2$ for all $a \in \Lambda$. There exists a continuous linear form sp on $\tau(\Lambda)$ (trace) satisfying $\text{sp}(ab) = \text{sp}(ba) = \langle a^*, b \rangle$. In particular, $\text{sp}(a^*a) = \text{sp}(aa^*) = \langle a, a \rangle = \|a\|^2 = \tau(a^*a)$.

A Hilbert Λ -module is a right module L over a proper H^* -algebra Λ provided with a mapping $[\cdot, \cdot]: L \times L \rightarrow \tau(\Lambda)$, which satisfies the following conditions: $[x, \alpha y] = \alpha[x, y]$; $[x, y + z] = [x, y] + [x, z]$; $[x, ya] = [x, y]a$; $[x, y]^* = [y, x]$; L is a Hilbert space with the inner product $\langle x, y \rangle = \text{sp}([x, y])$ for all $\alpha \in \mathbb{C}$, $x, y, z \in L$, $a \in \Lambda$ and for all $x \in L$, $x \neq 0$ there is $a \in \Lambda$, $a \neq 0$ such that $[x, x] = a^*a$.

A Λ -linear functional on L is a mapping $w': L \rightarrow \tau(\Lambda)$ such that $w'(xa + yb) = w'(x)a + w'(y)b$ for all $x, y \in L$, $a, b \in \Lambda$. It is bounded if there exists $c > 0$ such that $\tau(w'(x)) \leq c\|x\|$ for all $x \in L$. In this case we define $\|w'\| = \sup\{c \mid \tau(w'(x)) \leq c\|x\| \text{ for all } x \in L\}$. The norm space of all bounded Λ -linear functional f on L we denoted by L' .

Let $R(\Lambda)$ be the set of all bounded linear operators S on Λ such that $S(xy) = (Sx)y$ for all $x, y \in \Lambda$ and let $C(\Lambda)$ be the closed subspace of $R(\Lambda)$ generated by the operators of the form $La: x \rightarrow ax$, $a \in \Lambda$.

We now state some theorems which will be necessary for the proof of main results.

Theorem 2.1. [8, Theorem 3] *Each bounded Λ -linear functional f on L is of the form $[x_f, \cdot]$ for some $x_f \in L$.*

Theorem 2.2. [9, Lemma 1] *If $a \in \Lambda$ then the mapping $f_a: S \rightarrow \text{sp}(Sa)$, defined on $C(\Lambda)$, is a bounded linear functional and $\|f_a\| = \tau(a)$.*

Theorem 2.3. [9, Theorem 1] *Each bounded linear functional on $C(\Lambda)$ is of the form f_a for some $a \in \tau(\Lambda)$. The correspondence $a \leftrightarrow f_a$ is an isometric isomorphism between $\tau(\Lambda)$ and $C(\Lambda)^*$. Also, $\tau(\Lambda)$ is a Banach algebra.*

For more details, we refer to [8, 9, 1, 2, 5].

3. Results

Let L be a Hilbert H^* -module over a proper H^* -algebra Λ and let $B_1(L)$ is the unit ball in L . We construct a topology on W' such that the unit ball in L' is compact with respect to this topology. Define Λ' -weak* topology on L' with the base

$$L_{w'_0, x_1, \dots, x_n, S_1, \dots, S_n, \delta} = \{w' \in L' \mid |\text{sp}(S_j(w'(x_j) - w'_0(x_j)))| < \delta, j = 1, \dots, n\},$$

for $w'_0 \in L'$, $x_j \in L$, $S_j \in C(\Lambda)$, and $\delta > 0$.

The main result of this paper, the compactness of the unit ball, will be proven in Theorem 3.2 and its corollary. Before that, we state and prove a useful lemma.

Lemma 3.1. *Let Λ be a H^* -algebra and let $S \in C(\Lambda)$. Then the operator $S_a: \Lambda \rightarrow \Lambda$, $S_a(x) = S(ax)$ belongs to $C(\Lambda)$.*

Proof. From

$$S_a(\lambda x) = S(a\lambda x) = \lambda S(ax) = \lambda S_a(x), \quad S_a(xy) = S(axy) = S(ax)y = S_a(x)y$$

and

$$\|S_a(x)\| = \|S(ax)\| \leq \|S\| \cdot \|ax\| \leq \|S\| \cdot \|a\| \cdot \|x\|,$$

it follows that S_a belongs to $R(\Lambda)$. If L_{a_n} converges to S , then from

$$\begin{aligned} \|(L_{a_n} - S_a)(x)\| &= \|(L_{a_n}a(x) - S_a(x))\| = \|a_nax - S(ax)\| \\ &= \|L_{a_n}(ax) - S(ax)\| \leq \|L_{a_n} - S\| \cdot \|ax\| \\ &\leq \|L_{a_n} - S\| \cdot \|a\| \cdot \|x\|, \end{aligned}$$

it follows that $L_{a_n}a$ converges to S_a . Thus $S_a \in C(\Lambda)$. \square

Theorem 3.2. *The set*

$$K = \{w' \in L' \mid \|w'(x)\| \leq 1 \text{ for all } x \in B_1(L)\}$$

is compact in Λ' -weak topology.*

Proof. The neighborhood $B_1(L)$ is absorbing because for each $x \in L$, $x \neq 0$, there exists a number $\gamma(x) = \|x\| > 0$ such that $x \in \gamma(x)B_1(L)$. For all $w' \in K$, it holds $w'(x) \leq 1$, $x \in B_1(L)$, hence $|w'(x)| \leq \|x\|$, $x \in L$.

According to Banach-Alaoglu theorem and Theorem 2.3, the set $D_x = \{a \in \Lambda \mid \|a\| \leq \gamma(x)\}$ is compact in weak*-topology on $\tau(\Lambda)$ given by seminorms $p_S(\cdot) = |\text{sp}(S(\cdot))|$, $S \in C(\Lambda)$. Let τ_1 be the product weak*-topology on $P = \prod_x D_x$, the cartesian product of all D_x , one for each $x \in L$. Since each D_x is weak*-compact, it follows, from Tychonoff's theorem, that P is τ_1 -compact.

From definition of P we have

$$P = \{f: L \rightarrow \tau(\Lambda) \mid \|f(x)\| \leq \|x\| \text{ for all } x \in L\}.$$

The set P can contain Λ -nonlinear functionals.

It is clear that $K \subset L' \cap P$. It follows that K inherits two topologies: one from L' (its Λ' -weak* topology, to which the conclusion of the theorem refers) and the other, τ_1 , from P . We will see that these two topologies coincide on K , and that K is a closed subset of P .

We now prove that topologies τ_1 and Λ' -weak* coincide on K . Fix some $w'_0 \in K$. Then

$$\begin{aligned} W_1 = \{ w' \in L' \mid & |\text{sp}(S_j(w'(x_j))) - \text{sp}(S_j(w'_0(x_j)))| < \delta, \\ & x_1, x_2, \dots, x_n \in L, S_1, \dots, S_n \in C(\Lambda) \} \end{aligned}$$

and

$$W_2 = \{f \in P \mid |\text{sp}(S_j(f(x_j))) - \text{sp}(S_j(\Lambda_0(x_j)))| < \delta, \\ x_1, x_2, \dots, x_n \in L, S_1, \dots, S_n \in C(\Lambda)\}$$

are local bases in $(L', \Lambda\text{-weak}^*)$ and (P, τ_1) , respectively. From $K \subset L' \cap P$ we have $K \cap W_1 = K \cap W_2$, so topologies coincide on K .

Suppose w'_0 is in the τ_1 -closure of K . If S is from $C(\Lambda)$, then, from Lemma 3.1, operator $S(a \cdot): \Lambda \rightarrow \Lambda$ belongs to $C(\Lambda)$ for all $a \in \Lambda$. For any $a, b \in \Lambda$, $S \in C(\Lambda)$, $x, y \in L$, $\varepsilon > 0$, there is Λ -linear $w' \in K$ from $V_{xa+yb, w'_0, S, \varepsilon} \cap V_{x, w'_0, S(a \cdot), \varepsilon} \cap V_{y, w'_0, S(b \cdot), \varepsilon}$ (τ_1 -neighborhood of w'_0). Therefore, it holds

$$|\text{sp}(S[a(w'(x) - w'_0(x))])| < \varepsilon, \quad |\text{sp}(S[b(w'(y) - w'_0(y))])| < \varepsilon, \\ |\text{sp}(S[(w'(xa + yb) - w'_0(xa + yb))])| < \varepsilon.$$

We have

$$\begin{aligned} & |\text{sp}(S(w'_0(xa + yb) - w'_0(x)a - w'_0(y)b))| \\ &= |\text{sp}(S[(w'_0(xa + yb) - w'(xa + yb)) - (w'_0(x)a - w'(x)a \\ &\quad - (w'_0(y)b - w'(y)b))])| \\ &= |\text{sp}(S(w'_0(xa + yb) - w'(xa + yb)) - S[(w'_0(x) - w'(x))a] \\ &\quad - S[(w'_0(y) - w'(y))b])| \\ &\leq |\text{sp}(S(w'_0(xa + yb) - w'(xa + yb)))| + |\text{sp}(S[(w'_0(x) - w'(x))a])| \\ &\quad + |\text{sp}(S[(w'_0(y) - w'(y))b])| \\ &= |\text{sp}(S(w'_0(xa + yb) - w'(xa + yb)))| + |\text{sp}(aS[(w'_0(x) - w'(x))])| \\ &\quad + |\text{sp}(bS[(w'_0(y) - w'(y))])| \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we see that

$$\text{sp}(S(w'_0(xa + yb))) = \text{sp}(S(w'_0(x)a) + S(w'_0(y)b))$$

for all $S \in C(\Lambda)$, i.e.

$$\text{sp}(S(w'_0(xa + yb) - w'_0(x)a - w'_0(y)b)) = 0$$

for all $S \in C(\Lambda)$. For $S = L_{w'_0(xa+yb)-w'_0(x)a-w'_0(y)b}$ we have

$$\text{sp}((w'_0(xa + yb) - w'_0(x)a - w'_0(y)b))^*(w'_0(xa + yb) - w'_0(x)a - w'_0(y)b)) = 0.$$

Hence $w'_0(xa + yb) = w'_0(x)a + w'_0(y)b$ for all $x, y \in L$, $a, b \in \Lambda$, so w'_0 is Λ -linear.

Let $x \in B_1(L)$. For arbitrary $\varepsilon > 0$ and $S \in C_\Lambda$ there is $w' \in B_1(L')$ such that $|\text{sp}(S(w'_0(x))) - \text{sp}(S(w'(x)))| < \varepsilon$. Hence

$$|\text{sp}(S(w'_0(x)))| < |\text{sp}(S(w'(x)))| + \varepsilon \leq \|S\| \cdot \|w'(x)\| + \varepsilon \leq \|S\| + \varepsilon.$$

Next, from Theorem 2.2 we have

$$\begin{aligned} \|w'_0(x)\| &= \tau(w'_0(x)) = \|f_{w'_0(x)}\| = \sup_{S \in C(\Lambda), \|S\|=1} |\text{sp}(S(w'_0(x)))| \\ &\leq \sup_{S \in C(\Lambda), \|S\|=1} \|S\| + \varepsilon \leq 1 + \varepsilon \end{aligned}$$

for arbitrary $\varepsilon > 0$. Thus $\|w'_0(x)\| \leq 1$.

We have proven that $w'_0 \in B_1(L')$, and that $B_1(L')$ is a closed subset of P .

Since P is τ_1 -compact, $B_1(L')$ is a closed subset of P , and τ_1 and Λ' -weak* topology coincide on $B_1(L')$, we have that $B_1(L')$ is Λ' -weak* compact. \square

Corollary 3.3. *The unit ball $B_1(L)$ in L is compact in Λ -weak* topology with the base*

$W_{x_0, y_1, \dots, y_n, S_1, \dots, S_n, \delta} = \{x \in L \mid |\text{sp}(S_j([x, y_j] - [x_0, y_j]))| < \delta, j = 1, \dots, n\},$
for $x_0, y_j \in L, S_j \in C(\Lambda), \delta > 0$.

Proof. For each $w \in L'$ there is $y \in L$ such that $w'(x) = [y, x]$ for all $x \in L$ (Theorem 2.1), so from Theorem 3.2 it follows that the unit ball $B_1(L)$ in L is compact in given topology. \square

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